Ultrasound Image Reconstruction by Solving an Inverse Problem with Denoising Diffusion Restoration Models

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1/ Problem

Medical ultrasound images can be reconstructed by solving an inverse problem $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$, where

- $\mathbf{y} \in \Re^{K\times 1}$ gathers the sampled channel data ;
- $\mathbf{x} \in \Re^{N\times 1}$ contains the ultrasonic reflectivity values at N pixel positions ;
- $\mathbf{H} \in \Re^{K\times N}$ is the model matrix containing the information of geometry and the pulse-echo response ;
- $\mathbf{n} \in \Re^{K\times 1}$ stands for the white Gaussian noise with the standard deviation σ.

Traditional methods for solving the problem rely on using one or more regularization functions such as the l_1 norm, l_2 norm, or wavelet-based terms. However, finding a satisfactory balance between **image contrast**, **spatial resolution**, and **speckle preservation** is difficult. These features are crucial for medical ultrasound images.

2/ Basic Concepts

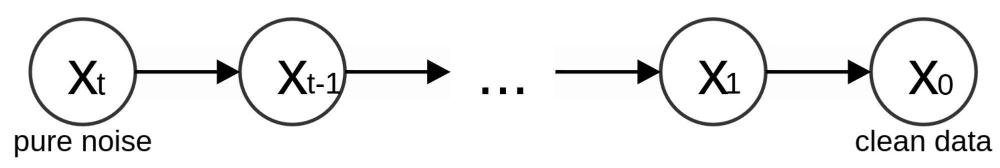

Figure 1 – The sampling process of Denoising Diffusion Probabilistic Models (DDPM)[3]

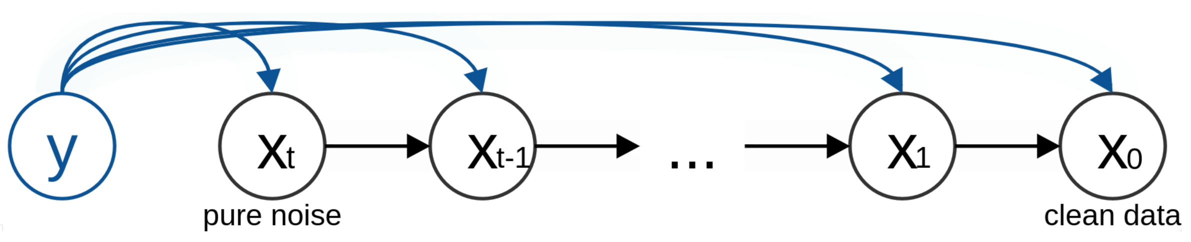

Figure 2 – The sampling process of Denoising Diffusion Restoration Models (DDRM)[2]

DDRM leverages $svd(\mathbf{H}_d)$ to make the sampling process dependent on the inverse problem :

$$\mathbf{y}_d = \mathbf{H}_d\mathbf{x}_d + \mathbf{n}_d \quad (1)$$

$$\mathbf{y}_d = \mathbf{U}_d\Sigma_d\mathbf{V}_d^t\mathbf{x}_d + \mathbf{n}_d \quad (2)$$

$$\Sigma_d^{\dagger}\mathbf{U}_d^t\mathbf{y}_d = \mathbf{V}_d^t\mathbf{x}_d + \Sigma_d^{\dagger}\mathbf{U}_d^t\mathbf{n}_d \quad (3)$$

$$\bar{\mathbf{y}}_d = \bar{\mathbf{x}}_d + \bar{\mathbf{n}}_d \quad (4)$$

Assuming that $\mathbf{n}_d \sim \mathcal{N}(0, \sigma_d{}^2)$. The singular values inside Σ_d are $s_1 > \ldots > s_i > \ldots > 0$, and

$$\bar{\mathbf{n}}_d \sim \mathcal{N}\left(0, \begin{bmatrix} \frac{\sigma_d^2}{s_1^2} & & & & \\ & \ddots & & & \\ & & \frac{\sigma_d^2}{s_i^2} & & \\ & & & \ddots & \\ & & & & \frac{\sigma_d^2}{s_N^2} \end{bmatrix}\right)$$

There exist three cases :

- $\frac{\sigma_d}{s_i} \leqslant \sigma_t$, then we believe $\bar{\mathbf{y}}_d$;
- $\frac{\sigma_d}{s_i} > \sigma_t$, then we assign weights to both $\bar{\mathbf{x}}_\theta$ and $\bar{\mathbf{y}}_d$;
- $\frac{\sigma_d}{s_i} = +\infty$, then we believe $\bar{\mathbf{x}}_\theta$. ($\bar{\mathbf{x}}_\theta = \mathbf{V}_d^t\mathbf{x}_\theta$, where $\mathbf{x}_\theta$ is generated from DDPM)

3/ Models

The traditional ultrasound inverse problem is

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

Because the size of $\mathbf{y}$ is always larger than that of $\mathbf{x}$. We use matched filtering to transform the inverse problem to

$$\mathbf{H}^t\mathbf{y} = \mathbf{H}^t\mathbf{H}\mathbf{x} + \mathbf{H}^t\mathbf{n} \quad (5)$$

And we consider a whitening operator $\mathbf{C} \in \Re^{N\times N}$ to further transform the inverse problem to

$$\mathbf{C}\mathbf{H}^t\mathbf{y} = \mathbf{C}\mathbf{H}^t\mathbf{H}\mathbf{x} + \mathbf{C}\mathbf{H}^t\mathbf{n}. \quad (6)$$

4/ C and $svd(\mathbf{H}_d)$

Given that $\mathbf{H}^t\mathbf{H}^*\mathbf{V} = \mathbf{V}^*\Lambda$,

for applying DDRM with Equ.5,

$$svd(\mathbf{H}_d) = svd(\mathbf{H}^t\mathbf{H}) = \mathbf{V}\Lambda\mathbf{V}^t, \; \sigma_d = \sigma(\mathbf{H}^t\mathbf{n}) ;$$

for applying DDRM with Equ.6,

$$\mathbf{C} = \Lambda^{\frac{1}{2}}\mathbf{V}^t,$$

$$svd(\mathbf{H}_d) = svd(\mathbf{C}\mathbf{H}^t\mathbf{H}) = \mathbf{I}\Lambda^{\frac{1}{2}}\mathbf{V}^t,$$

$$\sigma_d = \sigma(\mathbf{C}\mathbf{H}^t\mathbf{n}) = \sigma(\mathbf{n}).$$

5/ Simulation Results

Results with the phantom "fetus"[1] :

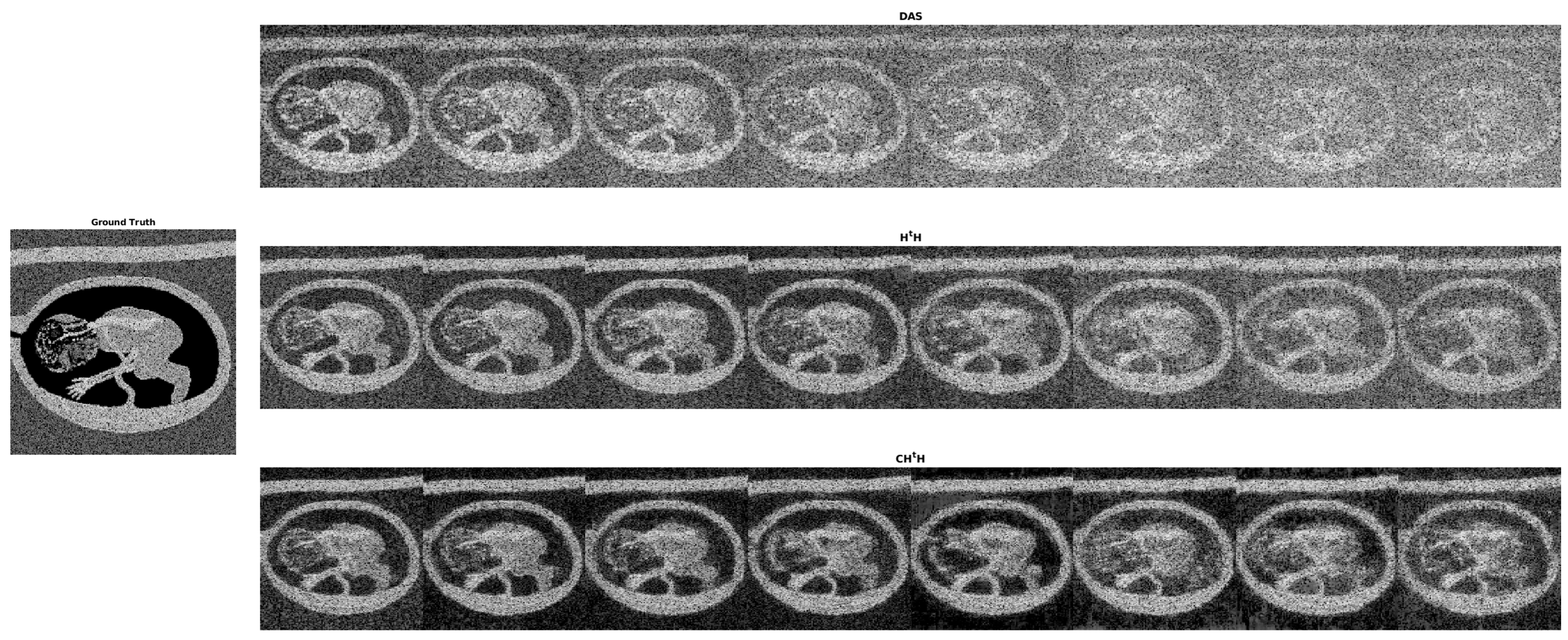

Figure 3 – The ground truth (left) and the restored images (right) reconstructed using delay-and-sum (top), models in Equ.5 (middle) and in Equ.6 (bottom). The standard deviation of the additive noise increases from left to right. All images are presented in dB units

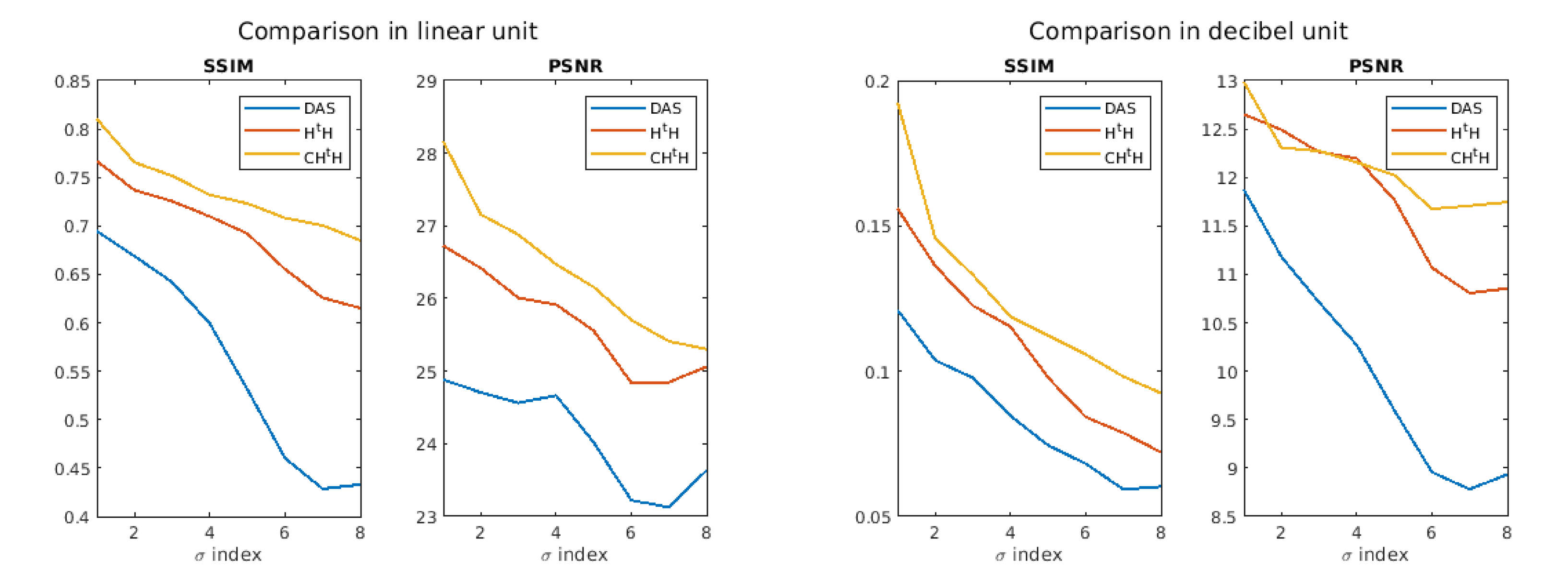

Figure 4 – Quantitative comparison using Structural SIMilarity (SSIM) and Peak Signal-to-Noise Ratio (PSNR)

6/ Conclusion

Summary

- The inverse problem model with a whitening operator is well suited to DDRM.
- The impact of the whitening operator is more pronounced with high-level additive noise.

Future Directions

- The current generative model is trained on natural images, and the next step is to train a new DDPM for medical ultrasound cases.

References

[1] Field II Ultrasound Simulation Program.

[2] Bahjat Kawar, Michael Elad, Stefano Ermon, and Jiaming Song. Denoising Diffusion Restoration Models, February 2022. arXiv :2201.11793 [cs, eess].

[3] Alex Nichol and Prafulla Dhariwal. Improved Denoising Diffusion Probabilistic Models, February 2021. arXiv :2102.09672 [cs, stat].